Spherical double flag varieties

Evgeny Smirnov

Higher School of Economics
Department of Mathematics

Independent University of Moscow

Torus Actions in Geometry, Topology, and Applications
Skolkovo, February 16, 2015

Outline

1. General definitions
   - Flag varieties
   - Schubert varieties
2. Spherical double flags of type A
   - Double Grassmannians
   - Combinatorics of B-orbits in double Grassmannians
3. Cominuscule flag varieties
   - Definition
   - Combinatorial and geometric results

Flag varieties

Notation

- G reductive algebraic group;

Flag varieties

Notation

- G reductive algebraic group;
- B fixed Borel subgroup;

Flag varieties

Notation

- G reductive algebraic group;
- B fixed Borel subgroup;
- T maximal torus contained in B;

Flag varieties

Notation

- G reductive algebraic group;
- B fixed Borel subgroup;
- T maximal torus contained in B;
- P parabolic subgroup, $P \supset B$;

Flag varieties

Notation

- G reductive algebraic group;
- B fixed Borel subgroup;
- T maximal torus contained in B;
- P parabolic subgroup, $P \supset B$;
- G/P (generalized) *partial flag variety*.

Flag varieties

Notation

- G reductive algebraic group;
- B fixed Borel subgroup;
- T maximal torus contained in B;
- P parabolic subgroup, $P \supset B$;
- G/P (generalized) *partial flag variety*.

Main example

- $G = GL(n)$;

Flag varieties

Notation

- G reductive algebraic group;
- B fixed Borel subgroup;
- T maximal torus contained in B;
- P parabolic subgroup, $P \supset B$;
- G/P (generalized) *partial flag variety*.

Main example

- $G = GL(n)$;
- B upper-triangular matrices;

Flag varieties

Notation

- G reductive algebraic group;
- B fixed Borel subgroup;
- T maximal torus contained in B;
- P parabolic subgroup, $P \supset B$;
- G/P (generalized) *partial flag variety*.

Main example

- $G = GL(n)$;
- B upper-triangular matrices;
- T diagonal matrices;

Flag varieties

Notation

- G reductive algebraic group;
- B fixed Borel subgroup;
- T maximal torus contained in B;
- P parabolic subgroup, $P \supset B$;
- G/P (generalized) *partial flag variety*.

Main example

- $G = GL(n)$;
- B upper-triangular matrices;
- T diagonal matrices;
- Two important subcases:

Flag varieties

Notation

- G reductive algebraic group;
- B fixed Borel subgroup;
- T maximal torus contained in B;
- P parabolic subgroup, $P \supset B$;
- G/P (generalized) *partial flag variety*.

Main example

- $G = GL(n)$;
- B upper-triangular matrices;
- T diagonal matrices;
- Two important subcases:
  - $P = B$: *full flag variety* $G/B = Fl(n)$.;

Flag varieties

Notation

- G reductive algebraic group;
- B fixed Borel subgroup;
- T maximal torus contained in B;
- P parabolic subgroup, $P \supset B$;
- G/P (generalized) *partial flag variety*.

Main example

- $G = GL(n)$;
- B upper-triangular matrices;
- T diagonal matrices;
- Two important subcases:
  - $P = B$: *full flag variety* $G/B = Fl(n)$.;
  - $P = P_{max}^{(k)}$: *Grassmannian* of k-planes $G/P_{max} = Gr(k, n)$.

Schubert decomposition

Schubert cells

- G/P has finitely many B-orbits Ω_w (G/P is a *spherical variety*);

Schubert decomposition

Schubert cells

- G/P has finitely many B-orbits Ω_w (G/P is a *spherical variety*);
- Each Ω_w has a unique T-stable point;

Schubert decomposition

Schubert cells

- G/P has finitely many B-orbits Ω_w (G/P is a *spherical variety*);
- Each Ω_w has a unique T-stable point;
- $G/P = \bigsqcup \Omega_w$ is a Białynicki-Birula decomposition;

Schubert decomposition

Schubert cells

- G/P has finitely many B-orbits Ω_w (G/P is a *spherical variety*);
- Each Ω_w has a unique T-stable point;
- $G/P = \bigsqcup \Omega_w$ is a Białynicki-Birula decomposition;
- Ω_w are indexed by the cosets W/W_I;

Schubert decomposition

Schubert cells

- G/P has finitely many B-orbits Ω_w (G/P is a *spherical variety*);
- Each Ω_w has a unique T-stable point;
- $G/P = \bigsqcup \Omega_w$ is a Białynicki-Birula decomposition;
- Ω_w are indexed by the cosets W/W_I;
- $X_w = \overline{\Omega_w}$ are called *Schubert varieties*.

Schubert decomposition

Schubert cells

- G/P has finitely many B-orbits Ω_w
  (G/P is a *spherical variety*);
- Each Ω_w has a unique T-stable point;
- $G/P = \bigsqcup \Omega_w$ is a Białynicki-Birula decomposition;
- Ω_w are indexed by the cosets W/W_I;
- $X_w = \overline{\Omega_w}$ are called *Schubert varieties*.

Schubert varieties

- Useful for enumerative geometry;

Schubert decomposition

Schubert cells

- G/P has finitely many B-orbits Ω_w (G/P is a *spherical variety*);
- Each Ω_w has a unique T-stable point;
- $G/P = \bigsqcup \Omega_w$ is a Białynicki-Birula decomposition;
- Ω_w are indexed by the cosets W/W_I;
- $X_w = \overline{\Omega_w}$ are called *Schubert varieties*.

Schubert varieties

- Useful for enumerative geometry;
- Have interesting geometry: they are usually singular...

Schubert decomposition

Schubert cells

- G/P has finitely many B-orbits Ω_w (G/P is a *spherical variety*);
- Each Ω_w has a unique T-stable point;
- $G/P = \bigsqcup \Omega_w$ is a Białynicki-Birula decomposition;
- Ω_w are indexed by the cosets W/W_I;
- $X_w = \overline{\Omega_w}$ are called *Schubert varieties*.

Schubert varieties

- Useful for enumerative geometry;
- Have interesting geometry: they are usually singular...
- ...however, still nice!

Schubert decomposition

Schubert cells

- G/P has finitely many B-orbits Ω_w (G/P is a *spherical variety*);
- Each Ω_w has a unique T-stable point;
- $G/P = \bigsqcup \Omega_w$ is a Białynicki-Birula decomposition;
- Ω_w are indexed by the cosets W/W_I;
- $X_w = \overline{\Omega_w}$ are called *Schubert varieties*.

Schubert varieties

- Useful for enumerative geometry;
- Have interesting geometry: they are usually singular...
- ...however, still nice!
  - normal;
  - Cohen–Macaulay;
  - have rational singularities...

Schubert varieties in Grassmannians

Let $U \in Gr(k, n)$, and let (V_i) be the flag fixed by B.

Schubert varieties in Grassmannians

Let $U \in Gr(k, n)$, and let (V_i) be the flag fixed by B.
Then $X_w \subset Gr(k, n)$ are given by conditions of type

$$\dim(U \cap V_i) \geq d_i, \qquad d_i - d_{i-1} \in \{0, 1\}, \quad d_n = k.$$

Schubert varieties in Grassmannians

Let $U \in Gr(k,n)$, and let (V_i) be the flag fixed by B.
Then $X_w \subset Gr(k,n)$ are given by conditions of type

$$\dim(U \cap V_i) \geq d_i, \qquad d_i - d_{i-1} \in \{0,1\}, \quad d_n = k.$$

Such sequences of 0's and 1's correspond to Young diagrams inside a $k \times (n-k)$-rectangle.

Schubert varieties in Grassmannians

Let $U \in Gr(k,n)$, and let (V_i) be the flag fixed by B.
Then $X_w \subset Gr(k,n)$ are given by conditions of type

$$\dim(U \cap V_i) \geq d_i, \qquad d_i - d_{i-1} \in \{0,1\}, \quad d_n = k.$$

Such sequences of 0's and 1's correspond to Young diagrams inside a $k \times (n-k)$-rectangle.

Example: $k = 4$, $n = 9$

$$d_i = (0 \quad 0 \quad 1 \quad 0 \quad 1 \quad 1 \quad 0 \quad 0 \quad 1)$$

Schubert varieties in Grassmannians

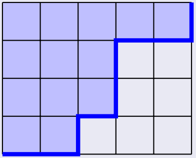
Let $U \in Gr(k,n)$, and let (V_i) be the flag fixed by B.
Then $X_w \subset Gr(k,n)$ are given by conditions of type

$$\dim(U \cap V_i) \geq d_i, \qquad d_i - d_{i-1} \in \{0,1\}, \quad d_n = k.$$

Such sequences of 0's and 1's correspond to Young diagrams inside a $k \times (n-k)$-rectangle.

Example: $k = 4$, $n = 9$

$$d_i = (0 \quad 0 \quad 1 \quad 0 \quad 1 \quad 1 \quad 0 \quad 0 \quad 1)$$



Spherical multiple flags

- Consider the direct product of several flag varieties:

Spherical multiple flags

- Consider the direct product of several flag varieties:

$$X = G/P_1 \times \cdots \times G/P_r.$$

Spherical multiple flags

- Consider the direct product of several flag varieties:

$$X = G/P_1 \times \cdots \times G/P_r.$$

- G acts diagonally.

Spherical multiple flags

- Consider the direct product of several flag varieties:

$$X = G/P_1 \times \cdots \times G/P_r.$$

- G acts diagonally.
- X is not G-homogeneous anymore!

Spherical multiple flags

- Consider the direct product of several flag varieties:
$$X = G/P_1 \times \cdots \times G/P_r.$$
- G acts diagonally.
- X is not G-homogeneous anymore!

Main question

When is X a G-spherical variety?
I.e., when does $B \subset G$ have a finite number of orbits on X?

Spherical multiple flags

- Consider the direct product of several flag varieties:

$$X = G/P_1 \times \cdots \times G/P_r.$$

- G acts diagonally.
- X is not G-homogeneous anymore!

Main question

When is X a G-spherical variety?
I.e., when does $B \subset G$ have a finite number of orbits on X?

Answer

- $r = 1$: always;

Spherical multiple flags

- Consider the direct product of several flag varieties:

$$X = G/P_1 \times \cdots \times G/P_r.$$

- G acts diagonally.
- X is not G-homogeneous anymore!

Main question

When is X a G-spherical variety?
I.e., when does $B \subset G$ have a finite number of orbits on X?

Answer

- $r = 1$: always;
- $r \geq 3$: never;

Spherical multiple flags

- Consider the direct product of several flag varieties:

$$X = G/P_1 \times \cdots \times G/P_r.$$

- G acts diagonally.
- X is not G-homogeneous anymore!

Main question

When is X a G-spherical variety?
I.e., when does $B \subset G$ have a finite number of orbits on X?

Answer

- $r = 1$: always;
- $r \geq 3$: never;
- $r = 2$: sometimes...

Spherical multiple flags

- Consider the direct product of several flag varieties:

$$X = G/P_1 \times \cdots \times G/P_r.$$

- G acts diagonally.
- X is not G-homogeneous anymore!

Main question

When is X a G-spherical variety?
I.e., when does $B \subset G$ have a finite number of orbits on X?

Answer

- $r = 1$: always;
- $r \geq 3$: never;
- $r = 2$: sometimes...
- ...classification given by P. Littelmann, J. Stembridge.

Type A: double Grassmannians

If $G = GL(n)$, all spherical double flag varieties correspond to P_1, P_2 maximal:

$$X = Gr(k,n) \times Gr(l,n).$$

Type A: double Grassmannians

If $G = GL(n)$, all spherical double flag varieties correspond to P_1, P_2 maximal:

$$X = Gr(k, n) \times Gr(l, n).$$

B-orbits correspond to configurations of two subspaces and a full flag, up to $GL(n)$-action.

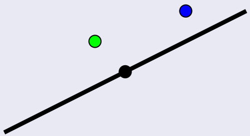
Type A: double Grassmannians

If $G = GL(n)$, all spherical double flag varieties correspond to P_1, P_2 maximal:

$$X = Gr(k, n) \times Gr(l, n).$$

B-orbits correspond to configurations of two subspaces and a full flag, up to $GL(n)$-action.
Combinatorics becomes more involved!

Example: $Gr(1,3) \times Gr(1,3) = \mathbb{P}^2 \times \mathbb{P}^2$



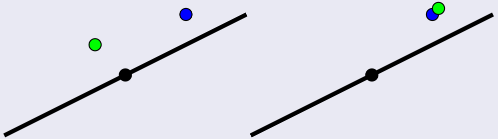
Type A: double Grassmannians

If $G = GL(n)$, all spherical double flag varieties correspond to P_1, P_2 maximal:

$$X = Gr(k, n) \times Gr(l, n).$$

B-orbits correspond to configurations of two subspaces and a full flag, up to $GL(n)$-action.
Combinatorics becomes more involved!

Example: $Gr(1,3) \times Gr(1,3) = \mathbb{P}^2 \times \mathbb{P}^2$



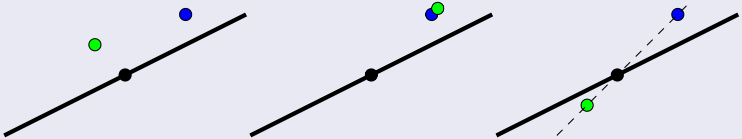
Type A: double Grassmannians

If $G = GL(n)$, all spherical double flag varieties correspond to P_1, P_2 maximal:

$$X = Gr(k, n) \times Gr(l, n).$$

B-orbits correspond to configurations of two subspaces and a full flag, up to $GL(n)$-action.
Combinatorics becomes more involved!

Example: $Gr(1,3) \times Gr(1,3) = \mathbb{P}^2 \times \mathbb{P}^2$



One $B \times B$-orbit splits into three B-orbits!

Combinatorics of B-orbits

- Our next goal is to describe B-orbits on $Gr(k, n) \times Gr(l, n)$.

Combinatorics of B-orbits

- Our next goal is to describe B-orbits on $Gr(k, n) \times Gr(l, n)$.
- $B \times B$-orbits are indexed by pairs of Young diagrams.

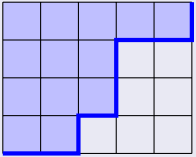
Combinatorics of B-orbits

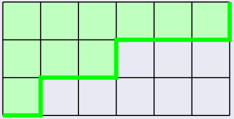
- Our next goal is to describe B-orbits on $Gr(k, n) \times Gr(l, n)$.
- $B \times B$-orbits are indexed by pairs of Young diagrams.
- But we need some extra data...

Combinatorics of B-orbits

- Our next goal is to describe B-orbits on $Gr(k, n) \times Gr(l, n)$.
- $B \times B$-orbits are indexed by pairs of Young diagrams.
- But we need some extra data...

The common diagram of two Young diagrams

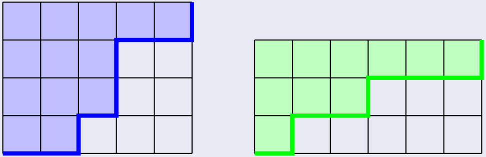




Combinatorics of B-orbits

- Our next goal is to describe B-orbits on $Gr(k, n) \times Gr(l, n)$.
- $B \times B$-orbits are indexed by pairs of Young diagrams.
- But we need some extra data...

The common diagram of two Young diagrams

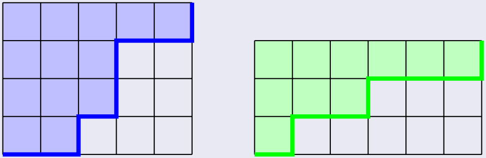


0	0	1	0	1	1	0	0	1
0	1	0	0	1	0	0	0	1
0			0	1		0	0	1

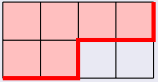
Combinatorics of B-orbits

- Our next goal is to describe B-orbits on $Gr(k,n) \times Gr(l,n)$.
- $B \times B$-orbits are indexed by pairs of Young diagrams.
- But we need some extra data...

The common diagram of two Young diagrams



0	0	1	0	1	1	0	0	1
0	1	0	0	1	0	0	0	1
0			0	1		0	0	1

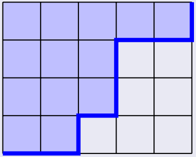


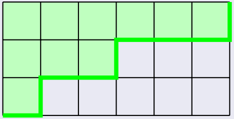
Consider *rook placements* in the common diagram.

Combinatorics of B-orbits

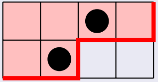
- Our next goal is to describe B-orbits on $Gr(k,n) \times Gr(l,n)$.
- $B \times B$-orbits are indexed by pairs of Young diagrams.
- But we need some extra data...

The common diagram of two Young diagrams





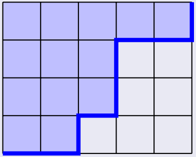
0	0	1	0	1	1	0	0	1
0	1	0	0	1	0	0	0	1
0			0	1		0	0	1

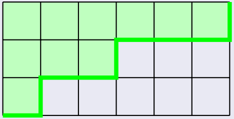


Combinatorics of B-orbits

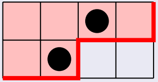
- Our next goal is to describe B-orbits on $Gr(k,n) \times Gr(l,n)$.
- $B \times B$-orbits are indexed by pairs of Young diagrams.
- But we need some extra data...

The common diagram of two Young diagrams





0	0	1	0	1	1	0	0	1
0	1	0	0	1	0	0	0	1
0			0	1		0	0	1



Consider *rook placements* in the common diagram.

Combinatorics of B-orbits

Theorem (S.)

B-orbits in $Gr(k, n) \times Gr(l, n)$ are indexed by triples (Y_1, Y_2, R), where:

Combinatorics of B-orbits

Theorem (S.)

B-orbits in $Gr(k, n) \times Gr(l, n)$ are indexed by triples (Y_1, Y_2, R), where:

- $Y_1 \subset k \times (n-k)$ and $Y_2 \subset l \times (n-l)$ are Young diagrams;

Combinatorics of B-orbits

Theorem (S.)

B-orbits in $Gr(k, n) \times Gr(l, n)$ are indexed by triples (Y_1, Y_2, R), where:

- $Y_1 \subset k \times (n-k)$ and $Y_2 \subset l \times (n-l)$ are Young diagrams;
- R is a rook placement in the common diagram of Y_1 and Y_2.

Combinatorics of B-orbits

Theorem (S.)

B-orbits in $Gr(k, n) \times Gr(l, n)$ are indexed by triples (Y_1, Y_2, R), where:

- $Y_1 \subset k \times (n-k)$ and $Y_2 \subset l \times (n-l)$ are Young diagrams;
- R is a rook placement in the common diagram of Y_1 and Y_2.

- Dimension and rank of orbits can be read from this description;

Combinatorics of B-orbits

Theorem (S.)

B-orbits in $Gr(k, n) \times Gr(l, n)$ are indexed by triples (Y_1, Y_2, R), where:

- $Y_1 \subset k \times (n-k)$ and $Y_2 \subset l \times (n-l)$ are Young diagrams;
- R is a rook placement in the common diagram of Y_1 and Y_2.

- Dimension and rank of orbits can be read from this description;
- This allows to construct resolutions of singularities of orbit closures à la Bott–Samelson–Demazure–Hansen.

Cominuscule double flags

Definition

A partial flag variety is *cominuscule* if it belongs to the following list:

Cominuscule double flags

Definition

A partial flag variety is *cominuscule* if it belongs to the following list:

Group type	Variety	
A_{n-1}	$Gr(k,n)$	Grassmannian
B_n	Q^{2n-1}	quadric
C_n	$LGr(n)$	Lagrangian Grassmannian
D_n	$OGr(n)$	orthogonal Grassmannian
	Q^{2n}	quadric
E_6	$\mathbb{OP}^2$	Cayley plane
E_7	$G_\omega(\mathbb{O}^3, \mathbb{O}^6)$	Lagrangian octonion Grassmannian

Cominuscule double flags

Definition

A partial flag variety is *cominuscule* if it belongs to the following list:

Group type	Variety	
A_{n-1}	$Gr(k,n)$	Grassmannian
B_n	Q^{2n-1}	quadric
C_n	$LGr(n)$	Lagrangian Grassmannian
D_n	$OGr(n)$ Q^{2n}	orthogonal Grassmannian quadric
E_6	$\mathbb{OP}^2$	Cayley plane
E_7	$G_\omega(\mathbb{O}^3,\mathbb{O}^6)$	Lagrangian octonion Grassmannian

We will consider *double cominuscule flag varieties*
(they are all spherical).

Results on cominuscule double flags

Combinatorics

- There is a combinatorial indexing of B-orbits in double cominuscule flag varieties of *classical groups*, similar to the one we had in the type A. (S.)

Results on cominuscule double flags

Combinatorics

- There is a combinatorial indexing of B-orbits in double cominuscule flag varieties of *classical groups*, similar to the one we had in the type A. (S.)
- Still open for E_6 and E_7.

Results on cominuscule double flags

Combinatorics

- There is a combinatorial indexing of B-orbits in double cominuscule flag varieties of *classical groups*, similar to the one we had in the type A. (S.)
- Still open for E_6 and E_7.

Geometry

- If G is simply laced, then the B-orbit closures in double cominuscule flag varieties are normal, Cohen–Macaulay, and have rational singularities (P.Achinger, N.Perrin, 2013).

Results on cominuscule double flags

Combinatorics

- There is a combinatorial indexing of B-orbits in double cominuscule flag varieties of *classical groups*, similar to the one we had in the type A. (S.)
- Still open for E_6 and E_7.

Geometry

- If G is simply laced, then the B-orbit closures in double cominuscule flag varieties are normal, Cohen–Macaulay, and have rational singularities (P.Achinger, N.Perrin, 2013).
- In type A was proved by methods of quiver theory (G.Bobiński, G.Zwara, 2001)

Results on cominuscule double flags

Combinatorics

- There is a combinatorial indexing of B-orbits in double cominuscule flag varieties of *classical groups*, similar to the one we had in the type A. (S.)
- Still open for E_6 and E_7.

Geometry

- If G is simply laced, then the B-orbit closures in double cominuscule flag varieties are normal, Cohen–Macaulay, and have rational singularities (P.Achinger, N.Perrin, 2013).
- In type A was proved by methods of quiver theory (G.Bobiński, G.Zwara, 2001)
- Normality can fail for nonsimply laced G.

That's all...

Thank you!